One dimensional strongly interacting Luttinger liquid of lattice spinless fermions*

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We consider the spinless fermion model with hard core repulsive potential between particles extended on a few lattice sites δ. The Luttinger liquid behavior is studied for the different values of a hard core radius. We derive a critical exponent Θ of the one particle correlation function $\langle c_i^\dagger c_j \rangle$ for an arbitrary electron density and coupling constant. Our results show that at the high density the behavior of fermions can be described as a strongly interacting Luttinger liquid with $\Theta > 1$. As a result, the residual Fermi surface disappears.

The anisotropic one dimensional (1D) S=$\frac{1}{2}$ Heisenberg model or the XXZ spin-chain is one of the simple integrable models of the condensed matter physics[1]. The conception of Luttinger liquid introduced by Haldane[2] is a very productive and useful for the description of one- dimensional systems[3] where power-law singularities in various correlators take place. The correlation lengths of spin correlation function and one-particle correlation function of spinless fermions at finite magnetic field and finite temperature have been calculated by means of a numerical study of the quantum transfer matrix[4]. Authors have obtained a nonanalytic temperature dependence of correlation length of the spin correlator $\langle \sigma_i^z \sigma_j^z \rangle$ and the Fermi momentum. They have found in this case a crossover temperature which depends on coupling constant and filling. Another scenario of a behavior of the one-particle correlation function has been obtained in the framework of the integrable one dimensional models with hard-core repulsive potential[5] . In Refs.[5] we discussed a strongly interacting Luttinger liquid state at high electron density characterized by a large value of critical exponent Θ for the momentum distribution function at T=0 ($\Theta > 1$). In this case the residual Fermi surface disappears. A critical density n_c ($\Theta(n_c) = 1$) separates a Luttinger liquid at a low electron density and an insulator phase that takes place at an extreme density $n_{\max} = \frac{1}{1+\delta}$. Note, that a strongly interacting Luttinger liquid state and a high-T$_c$ superconducting phase are realized at a small doping.

In this Letter we consider the behavior of spinless fermions in the framework of the extended version of the XXZ spin-chain proposed by Alcaras and Bariev recently[6]. As a concrete example we take the spinless fermion model with a hard core repulsive interaction. The model Hamiltonian contains a generalized projector that forbids configurations with two particles locating at distances less than or equal to δ. In this case the particles interact only when they are at the closest possible positions (i,i+δ +1, δ is measured in units of the lattice spacing parameter). The case $\delta = 0$ corresponds to the traditional XXZ spin-chain or the spinless fermion model. As we will see below, at high electron density the repulsive hard core potential leads to formation of a strongly interacting Luttinger liquid state without a residual Fermi surface. This example manifests a new many body state of 1D fermions, named as a strongly interacting Luttinger liquid[5]. As in Refs.[5], where this conception has been applied to different integrable models, we will discuss asymptotic behavior of the one-particle correlation function as a function of the electron density for several values of the coupling constant and the core radius. In the low temperature limit the critical exponents are evaluated of the scaling dimensions from finite size corrections to the energy spectra and can be calculated using the thermodynamic Bethe equations. The behavior of XXZ-spin chain depends on the value of an anisotropic exchange interaction Δ: in the absence of an external magnetic field the point $\Delta = 1$ separates the gap and gapless antiferromagnetic states. We shall consider both cases, namely $\Delta < 1$ and $\Delta > 1$, comparing a strongly interacting Luttinger liquid behavior in the insulator and metal states.

The model can be considered as a generalization of the well-known spinless fermion model with the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{L} \mathcal{P}_\delta (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + \Delta n_i n_{i+1+\delta}) \mathcal{P}_\delta, \qquad (1)$$

where $c_i^\dagger$ and c_i are spinless creation and annihilation operators of fermions at lattice site i, the hopping integral equal to unit and the coupling constant Δ is dimensionless, $\mathcal{P}_\delta$ is the projector forbidding two particles at distances less than or equal to δ. By $n_i = c_i^\dagger c_i$ we denote the number operator for particles on site i. The system consists of N particles on the chain with L sites (L is assumed to be even). The Hamiltonian (1) is transformed into XXZ-chain using Jordan-Wigner representation .

*Dedicated to Renat Bariev's memory

Then, model (1) was exactly solved by the Bethe ansatz method[6]. The two-particle scattering matrix is multiplied by an additional scattering phase shift due to the hard core potential

$$S(k_i,k_j)=\exp[-i\delta(k_i-k_j)]S_H(k_i,k_j), \tag{2}$$

where $S_H(k_i,k_j)$ is the two-particle scattering matrix of the XXZ spin-chain

$$S_H(k_i,k_j)=\frac{1+\exp i(k_i+k_j)+\Delta\exp ik_j}{1+\exp i(k_i+k_j)+\Delta\exp ik_i}. \tag{3}$$

At $\delta=0$ the two-particle scattering matrix (2) is reduced to the one for the traditional XXZ chain. In the present Letter we consider two repulsive critical regimes $0\leqslant\ \Delta\ <\ 1$ and $\Delta\ >\ 1$, introducing the parametrization for the coupling constant $\Delta=\cos\eta$ ($0<\eta\leqslant\frac{\pi}{2}$) and $\Delta=\cosh\eta$, respectively.

For $0\leqslant\ \Delta\ <\ 1$ the charge rapidities $\lambda_j(j=1,...,N)$ related to the momenta of particles k_j $\left(\exp(ik_j)=\frac{\sinh\frac{1}{2}(\lambda_j+i\eta)}{\sinh\frac{1}{2}(\lambda_j-i\eta)}\right)$ are obtained by solving the Bethe ansatz equations

$$\left(\frac{\sinh\frac{1}{2}(\lambda_j+i\eta)}{\sinh\frac{1}{2}(\lambda_j-i\eta)}\right)^{L-\delta N}=(-1)^N\exp(-i\delta P)\prod_{j=1}^{N}\frac{\sinh\frac{1}{2}(\lambda_j-\lambda_i+2i\eta)}{\sinh\frac{1}{2}(\lambda_j-\lambda_i-2i\eta)}, \tag{4}$$

where $P=\sum_{i=1}^{N}k_i$ is the momentum.

In terms of the rapidities λ_i the energy of the eigenvalues is given by

$$E=2N\cos\eta-2\sin^2\eta\sum_{j=1}^{N}\frac{1}{\cosh\lambda_j-\cos\eta}. \tag{5}$$

In the thermodynamic limit the ground state is described by the real rapidities $\{\lambda_j\}$. Taking the thermodynamic limit we obtain the integral equation of the Fredholm type for the distribution function $\rho(\lambda)$ for the variable λ,

$$\rho(\lambda)+\int_{-\Lambda}^{\Lambda}d\lambda' K_2^{<}(\lambda-\lambda')\rho(\lambda')=(1-\delta n)K_1^{<}(\lambda), \tag{6}$$

where the kernel $K_n^{<}(\lambda)$ is,

$$K_\nu^{<}(\lambda)=\frac{1}{2\pi}\frac{\sin(\nu\eta)}{\cosh\lambda-\cos(\nu\eta)}. \tag{7}$$

The λ-Fermi level denoted as Λ controls the band filling and the density of fermions is defined by

$$n=\int_{-\Lambda}^{\Lambda}d\lambda\ \rho(\lambda), \tag{8}$$

for $0\leqslant n\leqslant n_0$ and

$$1-(1+\delta)n=\int_{-\Lambda}^{\Lambda}d\lambda\ \rho(\lambda), \tag{9}$$

for $n_0\ <\ n\ \leqslant\ n_{\max}$. The particle density n_0 corresponds to a 'half-filling' $\frac{1}{2+\delta}$ and $n_{\max}$ is an extreme density that corresponds to the full band.

We remind the known results of the one-particle correlation function $\langle c^\dagger(x)c(0)\rangle$ at T=0 obtained in the framework of conformal field theory. The correlation function shows an oscillatory behavior and power-like decay with a scaling dimension Δ'

$$\langle c^\dagger(x)c(0)\rangle\simeq\cos(k_F x)x^{-2\Delta'}, \tag{10}$$

here k_F is the Fermi momentum. The momentum distribution function close to k_F is determined by the exponent Θ,

$$\langle n_k\rangle\simeq\langle n_{k_F}\rangle-const|k-k_F|^{\Theta}sgn(k-k_F), \tag{11}$$

where $\Theta=2\Delta'-1$ and

$$\Theta=\frac{1}{\alpha}\left(1-\frac{\alpha}{2}\right)^2. \tag{12}$$

The long-distance power-law behavior of the spin correlator $\langle\sigma_i^z\sigma_j^z\rangle$ is described by the critical exponent α; α=$2\zeta^2(\Lambda)$ is defined by the dressed charge $\zeta(\lambda)$ at the λ -Fermi level, $\zeta(\lambda)$ is the solution of the following integral equation

$$\zeta(\lambda)+\int_{-\Lambda}^{\Lambda}d\lambda' K_2^{<}(\lambda-\lambda')\zeta(\lambda')=(1-\delta n). \tag{13}$$

FIG. 1. Exponent α as a function of the electron density for $\eta=0.1$ (solid lines), $\pi/4$ (dashed); $\pi/3$ (dashed with points) and $\delta=0,1,2,3$. Result for $\delta=0$ is plotted (dotted line) for comparision.

FIG. 2. The exponent Θ , as in Fig. 1.The line separates the strongly interacting Luttinger liquid state.

The critical exponent α is analytically calculated for the densities $n=\ 0$ ($\alpha=2$), n_0 $\left(\alpha=4\frac{\pi}{\pi-\eta}n_0^2\right)$ and $n_{\max}$ $\left(\alpha=2n_{\max}^2\right)$. This was done using Wiener-Hopf method. Solving numerically the integral equations (6), (13) and taking into account conditions (8),(9) we show on Figs1 and 2 the critical exponents α and Θ as a function of the density for some values of Δ ($\eta=0,1;\pi/4;\pi/3$) and $\delta\ =\ 0,1,2,3$. The results of calculations obtained for the traditional XXZ chain ($\delta=0$) are plotted also for comparison in figures using dotted lines. In low density limit $n\to 0$ α converges to the value 2, which is the same as for non-interacting fermions. In the high density limit $n\to n_{\max}$ the value of α is independent on the coupling constant $\alpha=2n_{\max}^2$. In this case fermions are frozen and can be described by the second term in (1) . The model

Hamiltonian does not depend on the coupling constant (so, we can choose it equals unit) and, as a result, the values of the critical exponents α and Θ are independent on Δ (see Figs 3 and 4 also). For $\delta > 0$ a density of holes changes in the interval from $\delta/(1+\delta)$ to 1 whereas the density of fermions varies from $1/(1+\delta\ )$ to 0. The hard core potential makes essential changes in the structure of the hole state. Note that this interaction also destroys the hole-particle symmetry.

For $\delta > 0$ the function is non-symmetric with respect to 'half-filling' point and the value of α is less than 1 at $n > 0.5$. The correlation effects obtained due to the hard-core potential were most impressively displayed in the Θ behavior. According to (12) small values of the correlation exponent α leads to a large value of the Θ correlation exponent (see Fig.2). Comparing the behavior of Θ at $\delta = 0$ and $\delta \gneqq 0$ we observe an extremely large value of Θ in the high density region which reaches $\frac{1}{2}(n_{\max}^{-1} - n_{\max})^2$ at maximal possible density $n_{\max}$. Instead of the trivial value of $\Theta = 0$ that takes place at $\delta = 0$ we obtain $\frac{1}{2}\left(\frac{3}{2}\right)^2$ at $\delta = 1$, $\frac{1}{2}\left(\frac{8}{3}\right)^2$ at $\delta = 2$ and $\frac{1}{2}\left(\frac{15}{4}\right)^2$ at $\delta = 3$. The value of Θ increases with Δ and has a maximum value at a 'half-filling' for a large value of the coupling constant Δ. In Fig.2 we separate by a dotted line the region of density that corresponds to a strongly interacting Luttinger liquid state with $\Theta > 1$. The critical exponent of the one-particle correlation function (10) also has a large value in this region ($2\Delta' > 2$).

Using an analogous parametrization for the momenta $\exp(ik_j) = \frac{\sin\frac{1}{2}(\lambda_j + i\eta)}{\sin\frac{1}{2}(\lambda_j - i\eta)}$ we obtained the Bethe equations for another region of the interaction coupling $\Delta = \cosh\eta > 1$ in the form

$$\left(\frac{\sin\frac{1}{2}(\lambda_j + i\eta)}{\sin\frac{1}{2}(\lambda_j - i\eta)}\right)^{L-\delta N} = (-1)^N \exp(-i\delta P)$$

$$\prod_{j=1}^{N} \frac{\sin\frac{1}{2}(\lambda_j - \lambda_i + 2i\eta)}{\sin\frac{1}{2}(\lambda_j - \lambda_i - 2i\eta)}. \quad (14)$$

For the distribution function $\rho(\lambda)$ and the dressed charge $\zeta(\lambda)$ we have universal integral equations (6), (13) with a new kernel

$$K_\nu^{>}(\lambda) = \frac{1}{2\pi}\frac{\sinh(\nu\eta)}{\cosh(\nu\eta) - \cos(\lambda)}. \quad (15)$$

FIG. 3. Exponent α as a function of the electron density for $\eta = 0.1$ (solid lines), 0.5 (dashed); 1 (dashed with points) and $\delta = 0, 1, 2, 3$. Result for $\delta = 0$ is plotted (dotted line) for comparision.

FIG. 4. The exponent Θ , as in Fig. 3.The line separates the strongly interacting Luttinger liquid state.

In Figs 3,4 we show results of calculations of the correlation exponents as a function of the electron density for different coupling constants Δ and δ. The behavior of α is similar to the above case. Note that a minimal value of α is realized for an arbitrary coupling at a 'half-density'. A 'half filling' is a special point in one dimensional chains. As we note below, the gapless ground state for $\Delta < 1$ transforms to the gap antiferromagnetic state for $\Delta > 1$. In Fig 4 the Θ - cusps characterize a gap phase for $\Delta > 1$, a residual peak takes place for $\Delta < 1$ at small η only (see a solid line in Fig 2). This leads to the maximum of the value of Θ for arbitrary Δ and δ (see the curves obtained for $\eta = 0.1; 0.5; 1$ and $\delta = 0, 1, 2, 3$ in Fig.4). Θ is an increasing function of the parameters of the interactions Δ and δ. According to numerical calculation the critical density n_c ($\Theta(n_c) = 1$) is less then n_0. The value of n_c depends on the coupling constant Δ and the hard-core radius.

A 'half-filling' point is a singular one in the behavior of the critical exponents. When $\Delta > 1$ antiferromagnetic state with a gap takes place the Θ has a sharp maximum in this point. The width of the maximum is defined by a new scale, the gap $\Delta_c = 4\sinh\eta\frac{K}{\pi}k'$, where $\eta = \pi K'/K$, k and k' are modulus and complementary modulus of Jacobian elliptic functions and integrals, $K \equiv K(k)$ and $K' \equiv K(k')$ are complete and associated complete elliptic integrals of the first kind, respectively. The Δ_c trends to zero in the $\eta \to 0$ limit. The elliptic integrals K and K' are connected with coupling constant η and define a value of the gap. The gap in this state is a result of a large magnetic anisotropy. Thus, we can conclude that a strongly interacting Luttinger liquid is a state taking place near insulator or bad metal phases. It is possible also that a superconducting state has some features of a strongly interacting Luttinger liquid.

In summary, using an exact solution of generalized XXZ-spin model we extended the calculations of a strongly interacting Luttinger liquid state to the lattice spinless fermion system with a hard-core repulsive potential. The system has two qualitatively different regions. In the low-density regime where the hard core potential is not essential we obtain a traditional Luttinger liquid . In opposite high density limit when the hard-core potential dominates the state is a strongly interacting Luttinger liquid with a large value of the critical exponent $\Theta > 1$. This behavior has been illustrated by the numerical calculations. It is shown, that the value of the critical density that separates these regions depends on both δ and Δ. A main feature of this state is an absence of the residual Fermi surface.

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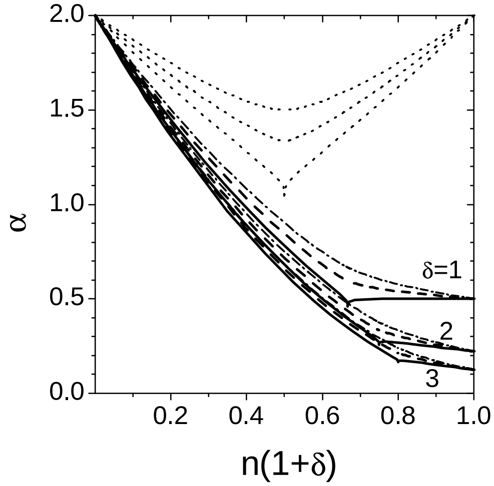
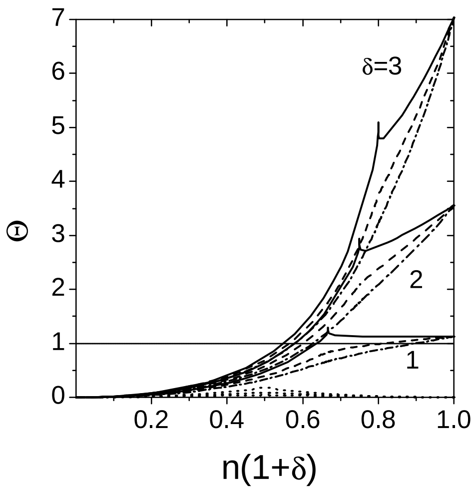
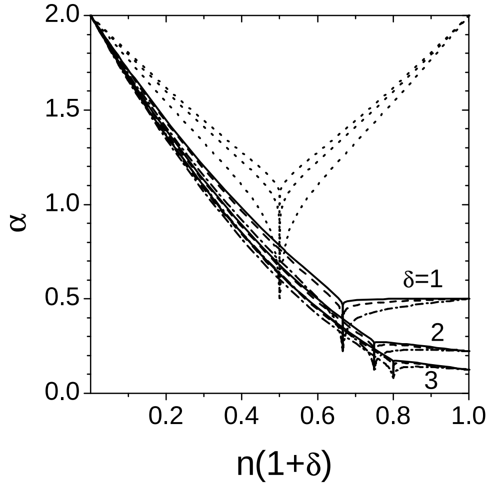
[1] C.N. Yang and C.P. Yang, Phys. Rev. **150**, 320,327 (1966).
[2] F.D.M. Haldane, J. Phys.C **14**, 2585 (1980); Phys. Rev. Lett. **45**, 1358 (1980).
[3] H.J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990); J. Voit, Rep. Prog. Phys. **58,** 977 (1995); J. Voit, Y. Wang, and M. Grioni, Phys. Rev. B **61**, 7930, 2000.
[4] K. Fabricius, A. Klümper, and B.M. McCoy, Phys. Rev. Lett. 82, 5365 (1999); K. Skai, M. Shroishi, J. Suzuki, and Y. Umeno, Phys. Rev. B 60, 5186 (1999); A. Klümper, J.R. Reyes Martinez, C. Scheeren, and M. Shiroishi, cond-mat/0004379.
[5] I.N. Karnaukhov, Phys. Rev. B 63, 45102 (2001); Europhys. Lett. 52, 571 (2000); I.N. Karnaukhov and N. Andrei (unpublished).
[6] F.C. Alcaraz and R.Z. Bariev, Phys. Rev. E 60, 79 (1999); cond-mat/9904042.



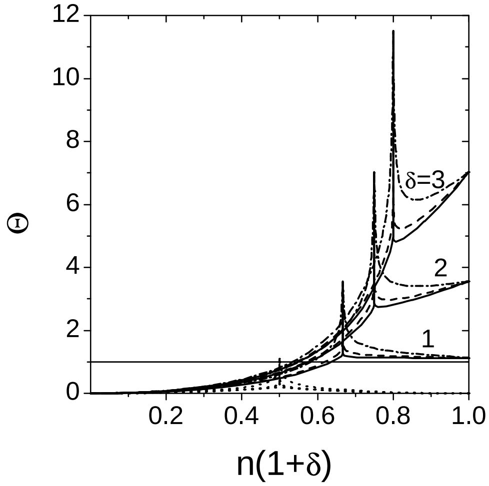
2.0
1.5
1.0
0.5
0.0
α
0.2
0.4
0.6
0.8
1.0
n(1+δ)
δ=1
2
3



δ=3
2
1
0
1
2
3
4
5
6
7
Θ
0.2
0.4
0.6
0.8
1.0
n(1+δ)



α
n(1+δ)
δ=1
2
3
0.0
0.5
1.0
1.5
2.0
0.2
0.4
0.6
0.8
1.0



12
10
8
6
4
2
0
Θ
0.2
0.4
0.6
0.8
1.0
n(1+δ)
δ=3
2
1